

# Field extension of real values of physical observables in classical theory can help attain quantum results

Hai Wang,<sup>1,\*</sup> Asutosh Kumar,<sup>2,3,†</sup> and Junde Wu<sup>1,‡</sup>

<sup>1</sup>*Department of Mathematics, Zhejiang University, Hangzhou 310027, PR China*

<sup>2</sup>*Harish-Chandra Research Institute, Chhatnag Road, Jhansi, Allahabad 211019, India*

<sup>3</sup>*Homi Bhabha National Institute, Anushaktinagar, Mumbai 400094, India*

Physical quantities are assumed to take real values, which stems from the fact that an usual measuring instrument that measures a physical observable always yields a real number. Here we consider the question of what would happen if physical observables are allowed to assume complex values. In this paper, we show that by allowing observables in the Bell inequality to take complex values, a classical physical theory can actually get the same upper bound of the Bell expression as quantum theory. Also, by extending the real field to the quaternionic field, we can puzzle out the GHZ problem using local hidden variable model. Furthermore, we try to build a new type of hidden-variable theory of a single qubit based on the result.

## I. INTRODUCTION

We have learnt from experience that measured variables yield real values. Therefore physical quantities are assumed to take real values because a typical measuring apparatus that measures a physical observable always yields a real number. In quantum theory, a physical observable is represented by a hermitian operator as it admits a special decomposition and has real eigenvalues. However, the real value of all physical quantities is an assumption only. Moreover, the real value of eigenvalues of some operator does not mean that it is necessarily hermitian. There are examples of non-hermitian operators which possess real eigenvalues under some symmetry conditions, namely the  $\mathcal{PT}$  symmetry [1–3]. We ask the following question: Can the outcomes of a physical observable be complex? It may be the limitation of a measuring apparatus that it measures only real values, or we haven't invented a way out to naturally measure

complex values. In general, given a non-hermitian operator, its average value in a quantum state is a complex number. In quantum theory, the concept of *weak measurement* was introduced by Aharonov *et al.* [4–6] to study the properties of a quantum system in pre- and postselected ensembles. In that formalism, the measurement of an observable yields a weak value of the observable with unusual properties. In general, the weak value is complex, and can take values outside the eigenspectrum of the observable. In recent years, weak values have found several applications [7–20]. One can see Ref. [21] for a recent review on weak measurements.

However, as a mathematical structure, unlike the real field, the field of complex numbers does not admit an *order relation*. Therefore, it is more difficult to compare “observables” having complex numbers as eigenvalues. One might, however, argue that complex numbers can be represented as pairs of real numbers, and measure them separately. More generally, one can measure quantities which are expressible as an array of real numbers. This is alright in classical mechanics. However, one must remember, as already noted by Dirac [22], that measuring an “observable” with complex

---

\* 3090101669@zju.edu.cn

† asukumar@hri.res.in

‡ wjd@zju.edu.cn

eigenvalues by somehow measuring separately its real and pure imaginary parts could lead to problems in the quantum formalism, as two observations may “interfere” with one another.

Inspite of having complex eigenvalues, non-hermitian operators have found several applications [23–33] in studying open quantum systems in nuclear physics [23] and quantum optics [24], to name a few. Regardless of how a complex value could be measured or what would be its interpretation, here we consider the question of what would happen if physical observables are allowed to assume complex values. Bender and Boettcher [1] showed that by allowing physical quantities to take complex values, classical particles can behave much like particles in quantum theory. In this paper, we show that by allowing observables in the Bell inequality to take complex values, a classical physical theory can actually get the same upper bound of the Bell expression, as in quantum theory. Moreover, using the local hidden variable model, and by extending the real field to the quaternionic field, we visit the GHZ paradox and show that there exists a non-empty set of 9-tuples of quaternions which correspond to negative eigenvalue of the GHZ argument.

## II. BELL INEQUALITY AND THE CORRESPONDING UPPER BOUND

In this section, we begin by briefly recalling the Bell inequality in the version due to Clauser, Horne, Shimony, and Holt (CHSH) [34]. We then find the upper bound on the Bell-CHSH expression when the observables in the expression are allowed to assume complex values.

### A. Bell inequality

Bell inequality is a mathematical inequality involving certain averages of correlations of measure-

ments, derived using the assumptions of locality<sup>2</sup> and realism. Bell showed that quantum mechanics violates this inequality. That is, quantum mechanics cannot be both local and realistic. The setting is as follows. There are two observers, Alice (A) and Bob (B). Each of them has two experimental settings:  $A(a)$  and  $A(a')$  for Alice, and  $B(b)$  and  $B(b')$  for Bob. And all these observables are dichotomic, taking real values  $\pm 1$ .

To explain possible randomness of measured values of an observable in experiments, we assume the existence of variables  $\lambda \in \Lambda$ , which determine the value of each quantity, and which may in principle remain hidden from observation with currently available measuring technology. Such variables are usually referred to as “hidden variables”. Each run of an experiment corresponds to a realization of the random variable,  $\lambda$ , with a probability distribution  $\rho(\lambda)$  for  $\lambda \in \Lambda$ . The correlation function of the experiment which involves the measurements of  $A(a)$  and  $B(b)$  at Alice and Bob’s laboratories respectively, is denoted as  $E(a,b)$ , and is defined as follows:

$$E(a,b) = \int d\lambda \rho(\lambda) A(a,\lambda) B(b,\lambda). \quad (\text{II.1})$$

Here,  $A(a,\lambda)$  denotes the value of the observable  $A(a)$  when  $A(a)$  is measured by Alice in the run of the experiment corresponding to  $\lambda \in \Lambda$ . Similarly for  $B(b,\lambda)$ .  $A(a,\lambda)$  and  $B(b,\lambda)$  are allowed to assume the real values  $+1$  or  $-1$ . Similar definitions as in (II.1) can be given for the other measurement settings. Then the Bell inequality can be expressed as

$$|E(a,b) - E(a,b')| + |E(a',b) + E(a',b')| \leq 2. \quad (\text{II.2})$$

This inequality is valid in any physical theory that is local and realistic, and where we have allowed the physical observables to assume the values  $\pm 1$ . We refer to the expression on the left-hand-side of (II.2) as the Bell expression. If in any theory, the Bell expression exceeds the value 2, the theory cannot be both local and realistic. Quantum mechanics happens to be such a theory.

## B. Real values to complex values

In a typical Bell experiment, all observables, when measured take real values  $\pm 1$ . Pushing this boundary, we now assume that these observables can take complex values. To be precise, the observable  $A(a)$  can take  $\pm e^{i\theta_1}$ ,  $\theta_1 \in [0, 2\pi)$ , for example. The hidden variable  $\lambda$  is employed as follows:

$$A(a, \lambda) = (-1)^{f_1(\lambda)} e^{i\theta_1}, A(a', \lambda) = (-1)^{f_3(\lambda)} e^{i\theta_3} \quad (\text{II.3})$$

$$B(b, \lambda) = (-1)^{f_2(\lambda)} e^{i\theta_2}, B(b', \lambda) = (-1)^{f_4(\lambda)} e^{i\theta_4},$$

where functions  $f_i(\lambda) = 0$  or  $1$ ,  $\forall i$ .

Following the original derivation of the Bell inequality, we have:

$$|E(a, b) - E(a, b')|$$

$$= \left| \int d\lambda \rho(\lambda) (-1)^{f_1(\lambda)} e^{i\theta_1} (-1)^{f_2(\lambda)} e^{i\theta_2} \right.$$

$$\left. - \int d\lambda \rho(\lambda) (-1)^{f_1(\lambda)} e^{i\theta_1} (-1)^{f_4(\lambda)} e^{i\theta_4} \right|$$

$$\leq \int d\lambda \rho(\lambda) |(-1)^{f_2(\lambda)} e^{i\theta_2} - (-1)^{f_4(\lambda)} e^{i\theta_4}|,$$

as  $|(-1)^{f_1(\lambda)} e^{i\theta_1}| = 1$ . And similarly,

$$|E(a', b) + E(a', b')|$$

$$\leq \int d\lambda \rho(\lambda) |(-1)^{f_2(\lambda)} e^{i\theta_2} + (-1)^{f_4(\lambda)} e^{i\theta_4}|.$$

Hence,

$$|E(a, b) - E(a, b')| + |E(a', b) + E(a', b')|$$

$$\leq |e^{i\theta_2} + e^{i\theta_4}| + |e^{i\theta_2} - e^{i\theta_4}| \leq 2\sqrt{2}. \quad (\text{II.4})$$

The inequality (II.4) is valid for the theory that is given in (II.3). The theory is both local and realistic, and assumes complex values of the measurement results of the observables  $A(a)$ ,  $A(a')$ ,  $B(b)$ , and  $B(b')$ . Interestingly, this upper bound coincides

with the upper bound of the Bell-CHSH inequality<sup>3</sup> (Tsirelson's bound [35]) in quantum theory. Thus, by allowing observables in a Bell experiment scenario to take complex values, a classical physical theory can actually get the same upper bound of the Bell expression as in quantum theory, where by "classical physical theory", we mean a theory that is both local and realistic.

Is this upper bound achievable classically? That is, is it possible to appropriately choose

$$\rho(\lambda), f_i(\lambda), \theta_i, \forall i$$

to make the inequality an equality. The answer is affirmative. For example, let  $\theta_2 = 0, \theta_4 = \pi/2, \theta_1 = 7\pi/4, \theta_3 = \pi/4$ . And let  $\rho(\lambda)$  be supported on those  $\lambda$  which satisfy  $f_2 = f_4 = 0$  or  $f_2 = f_4 = 1$ . Then it is easy to check that the upper bound can actually be achieved.

## III. THE GHZ ARGUMENT

In this section, by extending the real field to the quaternionic field, we employ the traditional local hidden variable (LHV) model to address the GHZ problem.

### A. Original GHZ argument

The GHZ argument [36] reads that the quantum state,  $|\Psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$ , satisfies

$$O_i |\Psi\rangle = + |\Psi\rangle, \quad (\text{III.1})$$

with  $i \in \{1, 2, 3\}$  and  $O_1 = \sigma_x^1 \otimes \sigma_y^2 \otimes \sigma_y^3 \equiv \sigma_x^1 \sigma_y^2 \sigma_y^3$ ,  $O_2 = \sigma_y^1 \sigma_x^2 \sigma_y^3$ ,  $O_3 = \sigma_y^1 \sigma_y^2 \sigma_x^3$ .

In the traditional LHV model, each observable is represented by its possible values, not by its operator. So the conditions (III.1) translate to:

$$S_x^1 S_y^2 S_y^3 = +1 = S_y^1 S_x^2 S_y^3 = S_y^1 S_y^2 S_x^3,$$

where  $S_{x,y}^{1,2,3} = \pm 1$ . Then we have

$$S_x^1 S_x^2 S_x^3 = (S_x^1 S_y^2 S_y^3)(S_y^1 S_x^2 S_y^3)(S_y^1 S_y^2 S_x^3) = +1. \quad (\text{III.2})$$

However, in quantum mechanics, we have  $\sigma_x^1 \sigma_x^2 \sigma_x^3 |\Psi\rangle = -|\Psi\rangle$ , which is in contradiction with Eq. (III.2). Thus, there is a conflict between quantum mechanics and classical theory which admits local realism.

### B. Quaternionic LHV model

In the quaternionic LHV model, we allow physical quantities to take quaternion values. Since  $S_{x,y}^{1,2,3} = \pm 1$ , the condition  $S_x^1 S_y^2 S_y^3 = +1$ , has four possibilities:

$$S_x^1 = +1, S_y^2 = +1, S_y^3 = +1 \quad (1)$$

$$S_x^1 = +1, S_y^2 = -1, S_y^3 = -1 \quad (2)$$

$$S_x^1 = -1, S_y^2 = +1, S_y^3 = -1 \quad (3)$$

$$S_x^1 = -1, S_y^2 = -1, S_y^3 = +1 \quad (4)$$

Firstly, we consider the possibility (1). It just means that the hidden variable  $\lambda$  dictates the following set:

$$S_1^{xyy} = \{(i, \pm j, \pm k) \times (\pm i, j, \pm k) \times (\pm i, j, \pm k)\}.$$

In the same way, the possibilities (2)-(4) yield the following sets:

$$S_2^{xyy} = \{(i, \pm j, \pm k) \times (\pm i, -j, \pm k) \times (\pm i, -j, \pm k)\},$$

$$S_3^{xyy} = \{(-i, \pm j, \pm k) \times (\pm i, j, \pm k) \times (\pm i, -j, \pm k)\},$$

$$S_4^{xyy} = \{(-i, \pm j, \pm k) \times (\pm i, -j, \pm k) \times (\pm i, j, \pm k)\}.$$

So the condition  $S_x^1 S_y^2 S_y^3 = +1$  corresponds to the union of these four sets,  $\bigcup_{i=1}^4 S_i^{xyy} \equiv S^{xyy}$ .

Similarly, the conditions  $S_y^1 S_x^2 S_y^3 = +1$  and  $S_y^1 S_y^2 S_x^3 = +1$ , separately correspond to sets  $\bigcup_{i=1}^4 S_i^{yxy} \equiv S^{yxy}$  and  $\bigcup_{i=1}^4 S_i^{yyx} \equiv S^{yyx}$  respectively. Now we take intersection of these sets. If we get an empty set, then our method will not solve the GHZ puzzle. If we get some non-empty set, then we have to check whether hidden variables  $\lambda$  corresponding to that set can satisfy  $S_x^1 S_x^2 S_x^3 = -1$ .

Interestingly, we get a non-empty set, which is the union of the following sets:

$$\{(i, j, \pm k) \times (i, j, \pm k) \times (i, j, \pm k)\},$$

$$\{(i, j, \pm k) \times (-i, -j, \pm k) \times (-i, -j, \pm k)\},$$

$$\{(-i, -j, \pm k) \times (i, j, \pm k) \times (-i, -j, \pm k)\},$$

$$\{(-i, -j, \pm k) \times (-i, -j, \pm k) \times (i, j, \pm k)\}.$$

For these sets (or, consequently  $\lambda$ ), we can build probability distributions (in other words, we can build a state since for traditional local hidden variable models, a state is equivalent to a probability distribution on hidden variables). However, note that for these sets, considering  $S_x^1$ ,  $S_x^2$  and  $S_x^3$ , we can always get:

$$S_x^1 S_x^2 S_x^3 = -i.$$

Compare this result with the quantum case

$$\sigma_x^1 \sigma_x^2 \sigma_x^3 |\Psi\rangle = -|\Psi\rangle.$$

Thus, if we extend the real field to the quaternionic field, then even in the LHV model there can be correlations as strong as in quantum mechanics, and can puzzle out the GHZ problem.

### IV. HIDDEN-VARIABLE THEORY OF A QUBIT

As we have seen in sections II and III, when we allow physical quantities to take complex (quaternion) values, classical physics can attain quantum

physical limit. Furthermore, when this complex-valued classical physics (we call complex-valued classical physics “pseudo-classical” physics) gets the maximum in the Bell inequality, the observables involved can share the same relation with the quantum case (just what the above example says). In this regard, we ask *is it possible to iterate the quantum theory in one-qubit case using this pseudo-classical approach?* Below we propose a new type of hidden-variable theory of a single qubit via complexification of physical quantities.

### A. Observables

The critical question is how to convert the hermitian operators on the Hilbert space to functions of hidden variables. We all know that the most important nontrivial observables in the one-qubit case are spins. The example of Bell inequality above has actually shown the way to make spin-operator a function of hidden variables. Concretely, when spins are in the same plane, say, the x-y plane, we can do as follows:

$$\sigma_x \Rightarrow f_x(\lambda) = \pm 1, \sigma_{\vec{\alpha}} \Rightarrow f_{\vec{\alpha}}(\lambda) = \pm e^{i\alpha},$$

when the vector  $\vec{\alpha}$  in the x-y plane has an angle  $\alpha$  with the x axis. It's the physical quantity that determines which value a system can take and it's the hidden variable  $\lambda$  that determines which value a system actually takes, which is just the core concept of the hidden-variable theory.

The change of  $\pm 1$  to antipodal points on the circle is natural, more or less. However, the difficulty arises, because spin-operators of the same plane have exhausted all antipodal points on the complex plane and yet all spin-operators of one qubit are apparently not on the same plane. How can we define functions for spin-operators on the different planes? Meanwhile, the standard quantum theory tells us that Pauli spin-operators  $\sigma_x, \sigma_y, \sigma_z$  have the same important place when we compare

them with each other. So, this equivalence should also be embodied in the way we use functions to replace them. This forces us to adopt a further step in complexification. That is, we should introduce quaternions into the pseudo-classical physics:

$$\sigma_x \Rightarrow f_x(\lambda) = \pm i, \sigma_y \Rightarrow f_y(\lambda) = \pm j, \sigma_z \Rightarrow f_z(\lambda) = \pm k.$$

Then for an arbitrary spin-operator  $\sigma_{\vec{r}} = \vec{r} \cdot \vec{\sigma} = \sum_{i=1}^3 r_i \sigma_i$  with an unit vector  $\vec{r} = (r_x, r_y, r_z)^T \in \mathbb{R}^3$ , what we do is the following:

$$\sigma_{\vec{r}} \Rightarrow f_{\vec{r}}(\lambda) = \pm (r_x i + r_y j + r_z k).$$

Apparently, the identity operator appearing in the formal quantum theory should be replaced by the function  $f(\lambda) \equiv 1$ . Now for any hermitian operator on  $\mathbb{C}^2$ , we can use a quaternion-valued function to replace it using the aboved method.

Since physical quantity can take quaternions in this pseudo-classical model, then why do measurements yield only real values? Consider the measurement of spin in the x direction, for example. We can imagine measurement as a fictitious rigid pole. Considering the sphere of unit ball in the i-j-k space, when we say that we are measuring the spin on the x direction, it may means that the fictitious pole is along the x direction. Only when the fictitious pole is on the same direction as the spin is, it can make the value of the corresponding function real. After all, it is an essential requirement in the standard quantum mechanics that the result of a measurement be real.

Then can this model tell us something new or explain things differently, taking quantum theory into account? Here is an interesting example. We all know that spins on different directions cannot be measured at the same time. Standard quantum mechanics tell us that it is because of the noncommutativity of those corresponding operators that we cannot measure them simultaneously. However, in our model, the explanation is quite apparent. It is that the fictitious rigid pole of the mea-

surement apparatus cannot be placed in different angles at the same time.

## B. States

When we talk about states in a hidden-variable theory, this means that we should give both the range of hidden variables and the probability distribution corresponding to each state in the standard theory. That is exactly what we do next.

For a one-qubit state, in the standard theory, we know that if we obtain expectations of  $\sigma_x$ ,  $\sigma_y$  and  $\sigma_z$ , then we can predict the result about any other observable. So, considering the equal importance of these three spin-operators, the measurable set of hidden-variable  $\lambda$  should be divided into eight

---

disjoint sets of the same measure, as shown in Table I. For example, set 1 consists of those  $\lambda$ , which satisfy

$$f_x(\lambda) = i, f_y(\lambda) = j, f_z(\lambda) = k,$$

and set 8 is consisted of those  $\lambda$ , which satisfy

$$f_x(\lambda) = -i, f_y(\lambda) = -j, f_z(\lambda) = -k.$$

So, in terms of the standard theory, we can assume that the range of  $\lambda$  is the set

$$\lambda \in \{1, 2, 3, 4, 5, 6, 7, 8\}.$$

Once this is done, we can give the probability distribution of each state of a qubit. Firstly, consider some simple instances.

$$|+_x\rangle\langle+_x| \Rightarrow P_{+_x}(\lambda) = \frac{1}{4}, \forall \lambda = 1, 2, 3, 4; \quad |-_x\rangle\langle-_x| \Rightarrow P_{-_x}(\lambda) = \frac{1}{4}, \forall \lambda = 5, 6, 7, 8$$

$$|+_y\rangle\langle+_y| \Rightarrow P_{+_y}(\lambda) = \frac{1}{4}, \forall \lambda = 1, 2, 5, 6; \quad |-_y\rangle\langle-_y| \Rightarrow P_{-_y}(\lambda) = \frac{1}{4}, \forall \lambda = 3, 4, 7, 8$$

$$|+_z\rangle\langle+_z| \Rightarrow P_{+_z}(\lambda) = \frac{1}{4}, \forall \lambda = 1, 3, 5, 7; \quad |-_z\rangle\langle-_z| \Rightarrow P_{-_z}(\lambda) = \frac{1}{4}, \forall \lambda = 2, 4, 6, 8$$

The reason we design this correspondence is that for spins on the x, y, z directions, if the state is an eigenvalue of one, then we could know nothing about the other two.

Then, for an arbitrary state, can we give its probability distribution? For a qubit, we know that an

---

arbitrary state can be expressed as:

$$\rho = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma}),$$

where  $\vec{r} = (r_x, r_y, r_z) \in R^3$  satisfies  $\|\vec{r}\| \leq 1$ .

At a first glance, it's natural to guess that the probability distribution of an arbitrary state  $\rho$  is (mapping  $\sigma_x = |+_x\rangle\langle+_x| - |-_x\rangle\langle-_x| \rightarrow P_{+_x} - P_{-_x}$ ):

$$P_\rho = \frac{1}{2}[1 + r_x(P_{+_x} - P_{-_x}) + r_y(P_{+_y} - P_{-_y}) + r_z(P_{+_z} - P_{-_z})].$$

But this is not normalised. The normalized probability distribution for arbitrary  $\rho$  is [37]:

7

$$P_\rho = \frac{1}{2}[1/4 + r_x(P_{+x} - P_{-x}) + r_y(P_{+y} - P_{-y}) + r_z(P_{+z} - P_{-z})].$$

With this probability distribution, using our method, it's quite easy to check that this kind of probability distribution allows us to get the same results about the three observables  $\sigma_x$ ,  $\sigma_y$  and  $\sigma_z$  as we get in the standard way. Since, for a qubit state, once we know its probability distributions about the three spins, then we can reconstruct the state, or in other words, we can predict its probability distribution about any observable. Hence, it's reasonable to expect that our probability distributions of hidden variables can make the same prediction as those made by the standard quantum theory, as these two different methods share a complete agreement on the three critical observables. Discussion about this will be put in section V.

set no. ( $\lambda$ )	$f_x(\lambda), f_y(\lambda), f_z(\lambda)$
1	i, j, k
2	i, j, -k
3	i, -j, k
4	i, -j, -k
5	-i, j, k
6	-i, j, -k
7	-i, -j, k
8	-i, -j, -k

TABLE I. Any observable  $A = aI + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z$ , of a qubit is a hermitian operator where  $a \in \mathbb{R}$  and  $(r_x, r_y, r_z) \in \mathbb{R}^3$ . In the view of hidden-variable theory, it seems that once the hidden variable  $\lambda$  has determined the values of  $\sigma_x$ ,  $\sigma_y$  and  $\sigma_z$ , the value of any other observable is determined. This inspires us to divide the range of  $\lambda$  into eight disjoint sets, depending on the values of  $\sigma_x \Rightarrow f_x(\lambda) = \pm i, \sigma_y \Rightarrow f_y(\lambda) = \pm j, \sigma_z \Rightarrow f_z(\lambda) = \pm k$ . Of course, these eight sets should have the same "area". This is why we can assume that the hidden variable  $\lambda$  is chosen in the set  $\{1, 2, 3, 4, 5, 6, 7, 8\}$ .

Until now, it seems that this method is quite good. But there is something unsatisfactory. If  $\vec{r} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})^T$ , then the corresponding probability distribution is somewhat annoying. Based on easy computation,

$$P_{\vec{r}}(\lambda = 8) = \frac{1}{8}(1 - \sqrt{3}) < 0.$$

Can we find reasonable explanation for this negative probability? We interpret this negative item as the augmentation of its "retroaction". What does "retroaction" mean? From Table I, because set 8 corresponds to those  $\lambda$ , which satisfy

$$\lambda \in \{\lambda \mid f_x(\lambda) = -i, f_y(\lambda) = -j, f_z(\lambda) = -k\}.$$

So set 8's "retroaction" is set 1.

In concrete terms, for an arbitrary quantum state  $\rho$ , its probability distribution must observe the following constraints:

$$P_\rho(m) + P_\rho(9 - m) = \frac{1}{4}, \forall m = 1, 2, 3, 4.$$

Following these constraints, the word "retroaction" may be appropriate. In this way, the negative probability causes no trouble.

## V. DISCUSSIONS

### A. Imperfection

Let us go back to the question left in section IV B. For our hidden-variable model, if we want our constructed probability distribution  $P_\rho$  for a given  $\rho$  to give the same prediction about any observable as those got in the orthodox way, then it means that: *For any hidden-variable  $\lambda \in \{1, 2, \dots, 8\}$  and any dichotomic valued function  $f_A(\lambda)$ , we can find an appropriate way to decide which value this function should*

take for each  $\lambda$ , where the subscript  $A$  is an arbitrary hermitian operator on  $\mathbb{C}^2$  showing that this function is the one corresponding to it in our hidden-variable model.

However, we are unable to address this particular question.

## B. Evolution

In the standard quantum theory, we know that unitary operators play a vital role in the description of states' evolution. Then, can we also describe states' evolution in our method? Noting that in our model, for the qubit case, we have attempted to reconstruct the theory on the basis of hidden variable  $\lambda \in \{1, 2, \dots, 8\}$ . So if we can give a description of evolution, then this description should be connected with number  $1, 2, \dots, 8$ . The way to describe the evolution is to use elements of the permutation group  $S_8$  of  $\{1, 2, \dots, 8\}$  and some of their convex combination. Following is the motivation for our guess.

In our model, we know that the state  $|+_x\rangle\langle+_x|$  corresponds to the probability distribution  $P_{+_x}(\lambda) = \frac{1}{4}, \forall 1 \leq \lambda \leq 4$ , and the state  $|-_x\rangle\langle-_x|$  corresponds to the probability distribution  $P_{-_x}(\lambda) = \frac{1}{4}, \forall 5 \leq \lambda \leq 8$ . Then we can describe the change from  $|+_x\rangle\langle+_x|$  to  $|-_x\rangle\langle-_x|$  as follows, using our language:

$$P_{-_x}(\lambda) = P_{+_x}(s\lambda), \forall 1 \leq \lambda \leq 8.$$

The  $s$  mentioned above is an element of the group  $S_8$ , which makes the exchange between  $i$  and  $i + 4$  for  $\forall 1 \leq \lambda \leq 4$ . This is the way the group  $S_8$  enters into our description of evolution. Then because of the infinite ways of evolution in quantum theory, convex combinations of elements in  $S_8$  come into

being, in terms of probability distributions' normalization and the finiteness of elements in  $S_8$ .

## VI. CONCLUSION

It seems that quantum theory is a complexification of the classical theory. It may be the strong mutual interaction between physical systems and the measurement apparatus that causes the information stored in the angles lost, which makes the classical theory classical.

In this paper, we investigated to what degree classical theory can behave like quantum theory, when we allow physical quantities to take complex (quaternion) values. Surprisingly enough, we found that in the case of the Bell-CHSH experiment, classical theory yields the same upper bound as the quantum theory. Moreover, by employing LHV model and by extending the real field to the quaternionic field, we can puzzle out the GHZ problem. These results motivated us to build the "crazy" hidden-variable theory of a single qubit. We believe that this "complexification" method will make useful and interesting contribution in future research.

## ACKNOWLEDGMENTS

AK is indebted to Ujjwal Sen for illuminating discussions and reading the manuscript, and acknowledges research fellowship from the Department of Atomic Energy, Government of India. J.W. is supported by the Natural Science Foundation of China (Grants No. 11171301, No. 10771191, and No. 11571307) and the Doctoral Programs Foundation of the Ministry of Education of China (Grant No. J20130061).

---

[1] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).

[2] C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. Lett. **89**, 270401 (2002).



- [3] D. C. Brody and E.-M. Graefe, Phys. Rev. Lett. **109**, 230405 (2012).
- [4] Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. **60**, 1351 (1988).
- [5] Y. Aharonov and A. Botero, Phys. Rev. A **72**, 052111 (2005).
- [6] Y. Aharonov and L. Vaidman, in *Time in Quantum Mechanics*, Lecture Notes in Physics, Vol. **734** (2008) pp. 399-447.
- [7] E. Sjöqvist, Physics Letters A **359**, 187 (2006).
- [8] Y. Aharonov, S. Popescu, and J. Tollaksen, Physics Today **63**, 27 (2010).
- [9] A. Cho, Science **333**, 690 (2011).
- [10] H. Wiseman, Physics Letters A **311**, 285 (2003).
- [11] R. Mir, J. S. Lundeen, M. W. Mitchell, A. M. Steinberg, J. L. Garretson, and H. M. Wiseman, New Journal of Physics **9**, 287 (2007).
- [12] J. S. Lundeen and A. M. Steinberg, Phys. Rev. Lett. **102**, 020404 (2009).
- [13] A. M. Steinberg, Phys. Rev. Lett. **74**, 2405 (1995).
- [14] A. M. Steinberg, Phys. Rev. A **52**, 32 (1995).
- [15] Y.-S. Kim, J.-C. Lee, O. Kwon, and Y.-H. Kim, Nat. Phys. **8**, 117 (2012).
- [16] U. Singh, U. Mishra, and H. S. Dhar, Annals of Physics **350**, 50 (2014).
- [17] P. C. W. Davies, Phys. Rev. A **79**, 032103 (2009).
- [18] J. S. Lundeen, B. Sutherland, A. Patel, C. Stewart, and C. Bamber, Nature **474**, 188 (2011).
- [19] J. S. Lundeen and C. Bamber, Phys. Rev. Lett. **108**, 070402 (2012).
- [20] H. F. Hofmann, Phys. Rev. A **83**, 022106 (2011).
- [21] J. Dressel, M. Malik, F. M. Miatto, A. N. Jordan, and R. W. Boyd, Rev. Mod. Phys. **86**, 307 (2014).
- [22] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, 1981).
- Dirac writes, *one might think one could measure a complex dynamical variable by measuring separately its real and pure imaginary parts. But this would involve two measurements or two observations, which would be all-right in classical mechanics, but would not do in quantum mechanics, where two observations in general interfere with one another—it is not in general permissible to consider that two observations can be made exactly simultaneously, and if they are made in quick succession the first will usually disturb the state of the system and introduce an indeterminacy that will affect the second.*
- [23] H. Feshbach, Annals of Physics **5**, 357 (1958).
- [24] M. B. Plenio and P. L. Knight, Rev. Mod. Phys. **70**, 101 (1998).
- [25] Y. Aharonov, S. Massar, S. Popescu, J. Tollaksen, and L. Vaidman, Phys. Rev. Lett. **77**, 983 (1996).
- [26] P. A. M. Dirac, Rev. Mod. Phys. **17**, 195 (1945).
- [27] S. Chaturvedi, E. Ercolessi, G. Marmo, G. Morandi, N. Mukunda, and R. Simon, Journal of Physics A: Mathematical and General **39**, 1405 (2006).
- [28] L. M. Johansen, Phys. Rev. A **76**, 012119 (2007).
- [29] C. Bamber and J. S. Lundeen, Phys. Rev. Lett. **112**, 070405 (2014).
- [30] J. E. Moyal, Mathematical Proceedings of the Cambridge Philosophical Society **45**, 99 (1949).
- [31] A. Di Lorenzo, Phys. Rev. Lett. **110**, 010404 (2013).
- [32] A. Matzkin, Journal of Physics A: Mathematical and Theoretical **45**, 444023 (2012).
- [33] A. K. Pati, U. Singh, and U. Sinha, Phys. Rev. A **92**, 052120 (2015).
- The authors have proposed an experimentally verifiable method to measure the complex expectation value of an arbitrary non-hermitian operator. They showed that the average of a non-hermitian operator is a complex multiple of the weak value of the positive-semidefinite part of the non-hermitian operator.
- [34] J. S. Bell, Physics **1**, 195 (1964); J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969); N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Rev. Mod. Phys. **86**, 419 (2014).
- [35] B. S. Cirel'son, Lett. Math. Phys. **4**, 93 (1980).
- [36] D. M. Greenberger, M. A. Horne, and A. Zeilinger, in *"Bell's Theorem, Quantum Theory, and Conceptions of the Universe"*, edited by M. Kafatos (Kluwer, Dordrecht 1989).
- [37]  $P_\rho = \frac{1}{2}[1/4 + r_x(P_{+x} - P_{-x}) + r_y(P_{+y} - P_{-y}) + r_z(P_{+z} - P_{-z})]$  is normalized in the sense that  $\sum_{\lambda=1}^8 P_\rho(\lambda) = 1$ . However, one may argue that  $P'_\rho = \frac{1}{8}[1 + r_x(P_{+x} - P_{-x}) + r_y(P_{+y} - P_{-y}) + r_z(P_{+z} - P_{-z})]$  is also normalized since  $\sum_{\lambda=1}^8 P'_\rho(\lambda) = 1$ . But it should be true for arbitrary  $\rho$ . For  $\rho = |+_x\rangle\langle+_x|$  ( $\vec{r} = (1, 0, 0)$ ), for example, while  $\sum_{\lambda=1}^4 P_\rho(\lambda) = 1$ ,  $\sum_{\lambda=1}^4 P'_\rho(\lambda) = \frac{5}{8} (< 1)$ .